

On the initial-value problem in the kinetic theory of gases

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(Received 11 February 1971 and in revised form 14 June 1971)

The relaxation of an initially non-uniform gas to equilibrium is studied within the framework of the kinetic theory of gases. The macroscopic gas properties are taken to depend on one spatial dimension as well as the time. The amplitude of the non-uniformity is assumed to be small with a length scale large compared with the mean free path, and the Krook model of the Boltzmann collision integral is employed.

By applying multi-time scale perturbation methods to this reduced problem, uniformly valid analytical solutions for the macroscopic velocity, density and temperature are obtained. The macroscopic equations appropriate to each stage of the relaxation process are obtained in a straightforward and unambiguous manner. The distribution function obtained is shown to be a re-expansion of the Chapman–Enskog solution of the Krook equation, with additional terms accounting for the relaxation of the initial conditions to a near equilibrium form. The results indicate that the uniformly valid first approximation to the macroscopic velocity, density and temperature can be obtained from the Navier–Stokes equations, but that no purely macroscopic set of equations will suffice for the determination of higher approximations.

1. Introduction

The initial-value problem in the kinetic theory of gases represents one of the oldest and most interesting problems in gasdynamics. The molecular motion is described statistically by a particle distribution function $f(\mathbf{x}, \mathbf{v}, t)$ defined so that the number of molecules in a volume element $d^3\mathbf{x}$ about the point \mathbf{x} in a velocity range $d^3\mathbf{v}$ about \mathbf{v} at an instant t is given by $f d^3\mathbf{x} d^3\mathbf{v}$. The distribution function satisfies Boltzmann's equation (Chapman & Cowling 1961). The general solution of the Boltzmann equation, given $f(\mathbf{x}, \mathbf{v}, t_0)$ at some initial instant t_0 , has never been obtained.

However, the problem simplifies considerably when the initial distribution of particles has a spatial length scale which is long compared with the mean free path (defined as the average distance that a particle will travel before colliding with another). In this case the motion is known to be governed by macroscopic equations, the initial conditions relaxing to a nearly equilibrium state (locally in space) after a few collisions. The macroscopic equations obtained depend upon the method of analysis used and the degree of accuracy required of the answer.

(See Grad (1967) for a survey of approaches to this and other questions in kinetic theory.)

In the present work three important simplifications to the small Knudsen number (the ratio of the mean free path to the relevant length scale) initial-value problem are made. First, the Boltzmann equation is replaced by the Krook statistical model (Bhatnagar, Gross & Krook 1954) as the governing equation for the distribution function. Second, the initial disturbance is assumed to be a small perturbation of a pre-existing equilibrium state. (The equations are *not* linearized, however.) Finally, the motion is assumed one-dimensional. The multi-time scale technique is then applied to the reduced problem. McCune, Morse & Sandri (1963) applied the method to the general initial-value problem for the Krook equation and obtained considerable insight into the relationship between many of the approaches referred to above. However, when the method is applied to the reduced problem, it is possible to obtain a uniformly valid solution for the macroscopic variables of interest (number density, macroscopic velocity and temperature). Moreover, the extent to which the macroscopic equations determine the motion emerges in an unambiguous way. In fact the one-dimensional macroscopic solutions themselves do not seem to have been obtained in the general problem shown below.

2. Mathematical formulation

The starting point of the analysis is the Krook kinetic model specialized to a one-dimensional unsteady flow:

$$\left. \begin{aligned} \frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} &= n\kappa(T)(\Phi - f), \\ \Phi &= n \left[\frac{m}{2\pi kT} \right]^{\frac{3}{2}} \exp \left[- (m/2kT) (\mathbf{v} - \mathbf{u})^2 \right]. \end{aligned} \right\} \quad (1)$$

Here $f(\mathbf{x}, \mathbf{v}, t)$ is the particle distribution function. The particles each have a mass m and no internal degrees of freedom. The macroscopic quantities appearing are number density n , velocity \mathbf{u} , and temperature T . They are defined as moments of f as follows:

$$\left. \begin{aligned} n &= \int f d^3\mathbf{v}, \\ n\mathbf{u} &= \int \mathbf{v} f d^3\mathbf{v}, \\ \frac{3}{2}nkT &= \int \frac{1}{2}m(\mathbf{v} - \mathbf{u})^2 f d^3\mathbf{v}. \end{aligned} \right\} \quad (2)$$

The gas has a microscopic collision frequency $n\kappa(T)$ which can be chosen to reproduce the viscosity of the gas whose motion is being modelled; k is Boltzmann's constant. For a one-dimensional motion it is sufficient to consider two components of velocity. Let the component in the direction of increasing x be u , and that in the plane of constant x (whose direction is determined by the initial condition) be w .

Now consider the initial-value problem. In the undisturbed state the gas is at rest with number density n_0 and temperature T_0 . In this state

$$\Phi = f = n_0 \left[\frac{m}{2\pi kT_0} \right]^{\frac{3}{2}} \exp \left[- (m/2kT_0) \mathbf{v}^2 \right] \equiv \Phi_0.$$

The gas is then disturbed so that the departure of f from Φ_0 is characterized by an amplitude parameter $\epsilon \ll 1$. The length scale associated with the disturbance is large compared with the mean free path $l = (kT_0/m)^{1/2}(n_0\kappa(T_0))^{-1}$. Let the length scale L be $l\delta^{-1}$; where $\delta \ll 1$ is the Knudsen number.

The appropriate non-dimensional variables are then

$$\begin{aligned} f &= n_0(m/kT_0)^{3/2}F(\boldsymbol{\xi}, y, s, \epsilon, \delta), \\ \Phi &= n_0(m/kT_0)^{3/2}\Psi, \quad \Psi = [N/(2\pi\Theta)^{3/2}] \exp[-(1/2\Theta)(\boldsymbol{\xi} - \mathbf{U})^2], \\ n &= n_0N(y, s, \epsilon, \delta), \quad T = T_0\Theta(y, s, \epsilon, \delta), \\ \mathbf{u} &\equiv (u, w, 0) = (kT_0/m)^{1/2}\{U, W, 0\} \equiv (kT_0/m)^{1/2}\mathbf{U}(y, s, \epsilon, \delta), \\ x &= (kT_0/m)^{1/2}[n_0\kappa(T_0)\delta]^{-1}y, \quad t = (n_0\kappa(T_0))^{-1}s, \\ n\kappa(T) &= n_0\kappa(T_0)\Omega(s, y, \epsilon, \delta), \\ \mathbf{v} &\equiv (v_x, v_y, v_z) = (kT_0/m)^{1/2}\{\xi, \eta, \zeta\} \equiv (kT_0/m)^{1/2}\boldsymbol{\xi}. \end{aligned}$$

Equations (1) and (2) now become

$$\left. \begin{aligned} \frac{\partial F}{\partial s} + \delta\xi \frac{\partial F}{\partial y} &= \Omega(\Psi - F), \\ N &= \int F d^3\boldsymbol{\xi}, \\ N(U, V, 0) &= \int F(\xi, \eta, \zeta) d^3\boldsymbol{\xi}, \\ \frac{3}{2}N\Theta &= \int \frac{1}{2}F(\boldsymbol{\xi} - \mathbf{U})^2 d^3\boldsymbol{\xi}. \end{aligned} \right\} \quad (3)$$

The macroscopic conservation laws are obtained by multiplying (3) by 1, ξ , η and $\frac{1}{2}(\boldsymbol{\xi} - \mathbf{U})^2$: then integrating over $\boldsymbol{\xi}$ we obtain in corresponding order

$$\left. \begin{aligned} \frac{\partial N}{\partial s} + \delta \frac{\partial}{\partial y}(NU) &= 0, \\ N \left(\frac{\partial U}{\partial s} + \delta U \frac{\partial U}{\partial y} \right) + \delta \frac{\partial P_v}{\partial y} &= 0, \\ N \left(\frac{\partial W}{\partial s} + \delta U \frac{\partial W}{\partial y} \right) + \delta \frac{\partial P_t}{\partial y} &= 0, \\ \frac{3}{2}N \left(\frac{\partial \Theta}{\partial s} + \delta U \frac{\partial \Theta}{\partial y} \right) + \delta \left(P_v \frac{\partial U}{\partial y} + P_t \frac{\partial W}{\partial y} + \frac{\partial Q}{\partial y} \right) &= 0. \end{aligned} \right\} \quad (4)$$

These equations are, of course, the conservation of mass, longitudinal and transverse momentum, and energy respectively. The normal and transverse components of the pressure tensor (P_v and P_t respectively) and the heat flux vector Q are given by

$$\left. \begin{aligned} P_v &= \int (\xi - U)^2 F d^3\boldsymbol{\xi}, \\ P_t &= \int (\xi - U)(\eta - V) F d^3\boldsymbol{\xi}, \\ Q &= \int \frac{1}{2}F(\boldsymbol{\xi} - \mathbf{U})^2 (\xi - U) d^3\boldsymbol{\xi}. \end{aligned} \right\} \quad (5)$$

The initial-value problem may now be posed as follows:

$$F = (2\pi)^{-3/2} \exp[-\frac{1}{2}\boldsymbol{\xi}^2] + \epsilon Z(\boldsymbol{\xi}, y) \quad \text{at } s = 0.$$

The first term is the dimensionless form of Φ_0 . The perturbation Z is assumed to vanish exponentially in $|\xi|$ for large $|\xi|$. Moreover, the disturbance is also assumed to be initially bounded spatially in a manner that will be made clear as the analysis progresses.

The problem contains two small parameters ϵ and δ , and a perturbation analysis is indicated. It is both convenient and constructive to consider δ in the form

$$\delta = \epsilon/\mathcal{R}.$$

The parameter \mathcal{R} will ultimately be seen to play the role of a Reynolds number in the macroscopic motion. When this choice for δ is made, the motion evolves on three different time scales. The kinetic equation clearly evolves in a non-trivial manner on the s scale. Macroscopic quantities must also evolve on a scale such that $\phi \equiv (\epsilon/\mathcal{R})s$ is $O(1)$ in magnitude since the initial conditions have a length scale $(\epsilon/\mathcal{R})^{-1}$ longer than the mean free path. Finally, there is a time scale such that $\tau \equiv (\epsilon^2/\mathcal{R})s$ is $O(1)$ in magnitude. It is on this time scale that the decay to an equilibrium state required by the irreversibility of the kinetic equation occurs. Non-linear effects will also occur on the τ scale.

The perturbation method appropriate to this type of problem is the multi-time scale technique (see e.g. Carrier & Pearson 1968 or Cole 1968). The dependence of F on its parameters is assumed in the form

$$F = F(\xi, y, s, \phi, \tau, \epsilon, \mathcal{R}).$$

Since ϕ and τ are linearly related to s , the time derivatives are transformed to

$$\frac{\partial}{\partial s} \rightarrow \frac{\partial}{\partial s} + \frac{\epsilon}{\mathcal{R}} \frac{\partial}{\partial \phi} + \frac{\epsilon^2}{\mathcal{R}} \frac{\partial}{\partial \tau}.$$

The initial conditions must now be applied at $s = \phi = \tau = 0$. With these modifications, F (and hence all moments of F) is expanded in a power series in ϵ , each of whose coefficients depends on all other parameters. The system of equations is rendered determinate by requiring that all perturbations are bounded and the first-order perturbation is the dominant term everywhere for any finite value of $|\xi|$. The expansion is thus assumed in the form

$$\left. \begin{aligned} F &= (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}\xi^2) + \sum_{n=1} \epsilon^n F^{(n)}(\xi, s, \phi, \tau, y), \\ \Psi &= (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}\xi^2) + \sum_{n=1} \epsilon^n \Psi^{(n)}(\xi, s, \phi, \tau, y), \\ U &= \sum_{n=1} \epsilon^n U^{(n)}(s, \phi, \tau, y), \quad P_v = 1 + \sum_{n=1} \epsilon^n P_v^{(n)}(s, \phi, \tau, y), \\ N &= 1 + \sum_{n=1} \epsilon^n N^{(n)}(s, \phi, \tau, y), \quad P_t = \sum_{n=1} \epsilon^n P_t^{(n)}(s, \phi, \tau, y), \\ \Theta &= 1 + \sum_{n=1} \epsilon^n \Theta^{(n)}(s, \phi, \tau, y), \quad Q = \sum_{n=1} \epsilon^n Q^{(n)}(s, \phi, \tau, y). \end{aligned} \right\} \quad (6)$$

The revised form of (3) and (4) is now

$$\left. \begin{aligned} \frac{\partial F}{\partial s} + \frac{\epsilon}{\mathcal{R}} \left(\frac{\partial F}{\partial \phi} + \xi \frac{\partial F}{\partial y} \right) + \frac{\epsilon^2}{\mathcal{R}} \frac{\partial F}{\partial \tau} &= \Omega(\Psi - F), \\ \frac{\partial N}{\partial s} + \frac{\epsilon}{\mathcal{R}} \left[\frac{\partial N}{\partial \phi} + \frac{\partial}{\partial y} (NU) \right] + \frac{\epsilon^2}{\mathcal{R}} \frac{\partial N}{\partial \tau} &= 0, \\ N \left[\frac{\partial U}{\partial s} + \frac{\epsilon}{\mathcal{R}} \left(\frac{\partial U}{\partial \phi} + U \frac{\partial U}{\partial y} \right) + \frac{\epsilon^2}{\mathcal{R}} \frac{\partial U}{\partial \tau} \right] + \frac{\epsilon}{\mathcal{R}} \frac{\partial P_v}{\partial y} &= 0, \\ N \left[\frac{\partial W}{\partial s} + \frac{\epsilon}{\mathcal{R}} \left(\frac{\partial W}{\partial \phi} + U \frac{\partial W}{\partial y} \right) + \frac{\epsilon^2}{\mathcal{R}} \frac{\partial W}{\partial \tau} \right] + \frac{\epsilon}{\mathcal{R}} \frac{\partial P_t}{\partial y} &= 0, \\ \frac{3}{2} N \left[\frac{\partial \Theta}{\partial s} + \frac{\epsilon}{\mathcal{R}} \left(\frac{\partial \Theta}{\partial \phi} + U \frac{\partial \Theta}{\partial y} \right) + \frac{\epsilon^2}{\mathcal{R}} \frac{\partial \Theta}{\partial \tau} \right] + \frac{\epsilon}{\mathcal{R}} \left(P_v \frac{\partial U}{\partial y} + P_t \frac{\partial W}{\partial y} + \frac{\partial Q}{\partial y} \right) &= 0. \end{aligned} \right\} \quad (7)$$

The perturbation equations are now obtained by the substitution of (6) in (7) and equating the coefficient of each power of ϵ to zero. The initial condition is

$$F^{(n)}(\xi, 0, 0, 0, y) = Z(\xi, y); \quad F^{(n)}(\xi, 0, 0, 0, y) = 0 \quad (n = 2, 3, \dots).$$

The initial conditions on all macroscopic quantities are obtained by taking the appropriate moments of $F(\xi, 0, 0, 0, y)$.

3. The early stages of the motion

The first-order equations in the expansion defined above are

$$\begin{aligned} \partial F^{(1)} / \partial s + F^{(1)} - \Psi^{(1)} &= 0, \\ \partial \{N^{(1)}, U^{(1)}, \Theta^{(1)}\} / \partial s &= 0, \\ \Psi^{(1)} &= (2\pi)^{-\frac{3}{2}} \exp\left(\frac{1}{2}\xi^2\right) \{N^{(1)} + \xi \cdot U^{(1)} + (\frac{1}{2}\xi^2 - \frac{3}{2})\Theta^{(1)}\}. \end{aligned}$$

The solutions are

$$\left. \begin{aligned} N^{(1)} &= N^{(1)}(\phi, \tau, y); \quad U^{(1)} = U^{(1)}(\Phi, \tau, y); \quad \Theta^{(1)} = \Theta^{(1)}(\phi, \tau, y); \\ F^{(1)} &= \Psi^{(1)} + e^{-s} \mathcal{F}^{(1)}(\phi, \tau, \xi, y). \end{aligned} \right\} \quad (8)$$

To a first approximation the velocity, temperature and density does not change from their initial values while the distribution function relaxes to a Gaussian form. $\mathcal{F}^{(1)}$ satisfies an evolution equation to be determined at a later stage in the analysis. The components of the pressure tensor and heat flux vector are

$$\left. \begin{aligned} P_v^{(1)} &= N^{(1)} + \Theta^{(1)} + e^{-s} \int \mathcal{F}^{(1)} \xi^2 d^3\xi, \\ P_t^{(1)} &= e^{-s} \int \xi \eta \mathcal{F}^{(1)} d^3\xi \quad Q^{(1)} = e^{-s} \int \frac{1}{2} \xi^2 \xi \mathcal{F}^{(1)} d^3\xi. \end{aligned} \right\} \quad (9)$$

We now consider the second approximation. The conservation laws are

$$\left. \begin{aligned} \frac{\partial N^{(2)}}{\partial s} + \frac{1}{\mathcal{R}} \left(\frac{\partial N^{(1)}}{\partial \phi} + \frac{\partial U^{(1)}}{\partial y} \right) &= 0, \\ \frac{\partial U^{(2)}}{\partial s} + \frac{1}{\mathcal{R}} \left(\frac{\partial U^{(1)}}{\partial \phi} + \frac{\partial P_v^{(1)}}{\partial y} \right) &= 0, \\ \frac{\partial W^{(2)}}{\partial s} + \frac{1}{\mathcal{R}} \left(\frac{\partial W^{(1)}}{\partial \phi} + \frac{\partial P_t^{(1)}}{\partial y} \right) &= 0, \\ \frac{3}{2} \frac{\partial \Theta^{(2)}}{\partial s} + \frac{1}{\mathcal{R}} \left(\frac{3}{2} \frac{\partial \Theta^{(1)}}{\partial \phi} + \frac{\partial U^{(1)}}{\partial y} + \frac{\partial Q^{(1)}}{\partial y} \right) &= 0. \end{aligned} \right\} \quad (10)$$

When (8) and (9) are substituted into (10), it is clear that the only way to avoid terms growing linearly with s in $N^{(2)}$, $U^{(2)}$ and $\Theta^{(2)}$ is to require the terms independent of s to vanish in each of (10).

$$\left. \begin{aligned} \partial N^{(1)}/\partial\phi + \partial U^{(1)}/\partial y &= 0, \\ \partial U^{(1)}/\partial\phi + \partial(N^{(1)} + \Theta^{(1)})/\partial y &= 0, \\ \frac{3}{2} \partial\Theta^{(1)}/\partial\phi + \partial U^{(1)}/\partial y &= 0, \\ \partial W^{(1)}/\partial\phi &= 0. \end{aligned} \right\} \tag{11}$$

The first three of (11) are the one-dimensional acoustic equations. The solutions are

$$\left. \begin{aligned} U^{(1)} &= \frac{1}{2}[J^{(1)}(r, \tau) - G^{(1)}(\sigma, \tau)], \\ \Theta^{(1)} &= \frac{2}{5}\{H^{(1)}(\tau, y) + \frac{1}{2}(\frac{5}{3})^{\frac{1}{2}}[J^{(1)}(r, \tau) + G^{(1)}(\sigma, \tau)]\}, \\ N^{(1)} &= \frac{2}{5}\{-H^{(1)}(\tau, y) + \frac{3}{4}(\frac{5}{3})^{\frac{1}{2}}[J^{(1)}(r, \tau) + G^{(1)}(\sigma, \tau)]\}, \\ r &= y - (\frac{5}{3})^{\frac{1}{2}}\phi; \quad \sigma = y + (\frac{5}{3})^{\frac{1}{2}}\phi. \end{aligned} \right\} \tag{12}$$

The results at this stage may be interpreted as follows. The distribution function loses all memory of its initial condition after a few collisions, relaxing to local thermodynamic equilibrium as represented by the linearized Gaussian distribution $\Psi^{(1)}$. The ‘conserved’ quantities N , U and Θ do not vary appreciably from their initial values during this period. On the time scale characterized by values of ϕ which are $O(1)$, the longitudinal motion is acoustic, while the transverse motion has not yet changed from its initial state. The ultimate evolution of the motion for a time scale characterized by non-zero values of τ will be considered next. The result will be a system of partial differential equations governing the evolution of $J^{(1)}$, $G^{(1)}$ and $H^{(1)}$.

4. The determination of the distribution function

To proceed further it is necessary to consider the behaviour of the second-order quantities in some detail. From (9), (10) and (11)

$$\left. \begin{aligned} N^{(2)} &= N^{(2)}(\phi, \tau, y), \\ U^{(2)} &= U_M^{(2)}(\phi, \tau, y) - \frac{e^{-s}}{\mathcal{R}} \int \frac{\partial \mathcal{F}^{(1)}}{\partial y} \xi^2 d^3\xi, \\ W^{(2)} &= -\frac{e^{-s}}{\mathcal{R}} \int \xi \eta \frac{\partial \mathcal{F}^{(1)}}{\partial y} d^3\xi + W_M^{(2)}(\phi, \tau, y), \\ \Theta^{(2)} &= -\frac{2}{3\mathcal{R}} e^{-s} \int \frac{1}{2} \xi^2 \xi \frac{\partial \mathcal{F}^{(1)}}{\partial y} d^3\xi + \Theta_M^{(2)}(\phi, \tau, y). \end{aligned} \right\} \tag{13}$$

The subscript M denotes the part of each variable which evolves solely on macroscopic time scales. The second-order kinetic equation is

$$\begin{aligned} \frac{\partial F^{(2)}}{\partial s} + F^{(2)} &= \Psi^{(2)} + \Omega^{(1)}(\Psi^{(1)} - F^{(1)}) - \frac{1}{\mathcal{R}} \left(\frac{\partial F^{(1)}}{\partial\phi} + \xi \frac{\partial F^{(1)}}{\partial y} \right), \\ \Omega^{(1)} &\equiv N^{(1)} + \frac{1}{\kappa(T_0)} \frac{d\kappa}{dT_0}(T_0) \Theta^{(1)}, \end{aligned}$$

$$\Psi^{(2)} = (2\pi)^{-\frac{3}{2}} \exp -\frac{1}{2}\xi^2 \{N^{(2)} + \xi \cdot U^{(2)} + (\frac{1}{2}\xi^2 - \frac{3}{2}) \Theta^{(2)} - \xi \cdot \mathbf{U}^{(1)} \Theta^{(1)} \\ + \frac{1}{2}[N^{(1)} + \xi \cdot \mathbf{U}^{(1)} + (\frac{1}{2}\xi^2 - \frac{3}{2}) \Theta^{(1)}]^2 - \frac{1}{2}[N^{(1)2} + \mathbf{U}^{(1)2} + (\xi^2 - \frac{3}{2}) \Theta^{(1)2}]\}.$$

Using (8), (11) and (13) this can be written as

$$\left. \begin{aligned} \frac{\partial F^{(2)}}{\partial s} + F^{(2)} &= \Psi_M^{(2)} - \frac{1}{\mathcal{R}} (2\pi)^{-\frac{3}{2}} \exp(-\frac{1}{2}\xi^2) \left\{ (\xi^2 - \frac{1}{3}\xi^2) \frac{\partial U^{(1)}}{\partial y} + \xi \eta \frac{\partial W^{(1)}}{\partial y} \right. \\ &\quad \left. + \xi(\frac{1}{2}\xi^2 - \frac{5}{2}) \frac{\partial \Theta^{(1)}}{\partial y} \right\} - \frac{e^{-s}}{\mathcal{R}} S(\xi, \phi, \tau, y), \\ \Psi_M^{(2)} &\equiv \Psi^{(2)} + \frac{e^{-s}}{\mathcal{R}} (2\pi)^{-\frac{3}{2}} \exp(-\frac{1}{2}\xi^2) \left\{ \xi \int \frac{\partial \mathcal{F}^{(1)}}{\partial y} \xi' \xi' d^3\xi' \right. \\ &\quad \left. + (\frac{1}{2}\xi^2 - \frac{3}{2}) \int \frac{1}{3} \frac{\partial \mathcal{F}^{(1)}}{\partial y} (\xi')^2 \xi' d^3\xi' \right\}, \\ S &\equiv \frac{\partial \mathcal{F}^{(1)}}{\partial \phi} + \xi \frac{\partial \mathcal{F}^{(1)}}{\partial y} + \mathcal{R} \Omega^{(1)} \mathcal{F}^{(1)} + (2\pi)^{-\frac{3}{2}} \exp(-\frac{1}{2}\xi^2) \\ &\quad \times \left\{ \xi \int \xi' \xi' \frac{\partial \mathcal{F}^{(1)}}{\partial y} d^3\xi' + (\frac{1}{2}\xi^2 - \frac{3}{2}) \int \frac{1}{3} (\xi')^2 \xi' \frac{\partial \mathcal{F}^{(1)}}{\partial y} d^3\xi' \right\}. \end{aligned} \right\} \quad (14)$$

We first consider (14) as a function of s . Unless $S = 0$, $F^{(2)}$ will contain a term proportional to $(s/\mathcal{R}) e^{-s}$. Thus, when s is $O(\mathcal{R}/\epsilon)$ (i.e. when ϕ is $O(1)$) $(\epsilon/\mathcal{R}) \partial F^{(2)}/\partial s$ will be as large as $\partial F^{(1)}/\partial s$. Avoiding this breakdown in the perturbation scheme then produces the evolution equation for $\mathcal{F}^{(1)}$:

$$\frac{\partial \mathcal{F}^{(1)}}{\partial \phi} + \xi \frac{\partial \mathcal{F}^{(1)}}{\partial y} + \mathcal{R} \Omega^{(1)} \mathcal{F}^{(1)} = -(2\pi)^{-\frac{3}{2}} \exp(-\frac{1}{2}\xi^2) \left\{ \xi \int \xi' \xi' \frac{\partial \mathcal{F}^{(1)}}{\partial y} d^3\xi' \right. \\ \left. + (\frac{1}{2}\xi^2 - \frac{3}{2}) \int \frac{1}{3} (\xi')^2 \xi' \frac{\partial \mathcal{F}^{(1)}}{\partial y} d^3\xi' \right\}. \quad (15)$$

It is easy to show that $\mathcal{F}^{(1)}$ has the required property that

$$\int \mathcal{F}^{(1)}(1, \xi, \frac{1}{2}\xi^2) d^3\xi = 0.$$

Let the appropriate moments be denoted by

$$\int \mathcal{F}^{(1)}(1, \xi, \frac{1}{2}\xi^2) d^3\xi \equiv [n, \mathbf{u}, \theta].$$

Taking the corresponding moments of (15) gives

$$\begin{aligned} \partial n / \partial \phi + \mathcal{R} \Omega^{(1)} n + \partial u / \partial y &= 0, \\ \partial \mathbf{u} / \partial \phi + \mathcal{R} \Omega^{(1)} \mathbf{u} &= 0, \\ \partial \theta / \partial \phi + \mathcal{R} \Omega^{(1)} \theta &= 0. \end{aligned}$$

The last two of these equations may be formally solved

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_0(\tau, y) \exp \left[-\mathcal{R} \int_0^\phi \Omega^{(1)} d\phi \right], \\ \theta &= \theta_0(\tau, y) \exp \left[-\mathcal{R} \int_0^\phi \Omega^{(1)} d\phi \right], \\ \mathbf{u}_0(0, y) &= 0; \quad \theta_0(0, y) = 0. \end{aligned}$$

If \mathbf{u}_0 and θ_0 are independent of τ , then \mathbf{u} and θ are zero; whence

$$n = n_0(\tau, y) \exp \left[-\mathcal{R} \int_0^\phi \Omega^{(1)} d\phi \right]; \quad n_0(0, y) = 0.$$

Thus n vanishes also if n_0 is independent of τ . The dependence on τ of the quantities n_0 , \mathbf{u}_0 , and θ_0 is determined by the homogeneous solution $\mathcal{F}_H^{(1)}$ of (15)

$$\begin{aligned} \mathcal{F}_H^{(1)} = A^{(1)}(\xi, \tau y - \xi\phi) \exp \left\{ -\mathcal{R} \int_0^\phi \Omega^{(1)}(\theta', \tau, y + \xi(\phi' - \phi)) d\phi' \right\} \\ - A^{(1)}(\xi, 0, y) - Z(\xi, y) - \Psi^{(1)}(\xi, 0, 0, y). \end{aligned}$$

It is shown in the appendix that $A^{(1)}$ may be chosen to be independent of τ without introducing any inconsistencies in the perturbation scheme. This choice leads to \mathbf{u}_0 , n_0 and θ_0 which are independent of τ and thus zero. Since the primary purpose of this paper is to determine the behaviour of $N^{(1)}$, $\mathbf{U}^{(1)}$ and $\Theta^{(1)}$, the properties of $\mathcal{F}^{(1)}$ will not be pursued further here.

The solution for $F^{(2)}$ may now be written as

$$\begin{aligned} F^{(2)} = e^{-s} \mathcal{F}^{(2)}(\xi, \phi, \tau, y) + \Psi_M^{(2)} \\ - \frac{1}{\mathcal{R}} (2\pi)^{-\frac{1}{2}} \exp \left(-\frac{1}{2} \xi^2 \right) \left\{ \left(\xi^2 - \frac{1}{3} \xi^2 \right) \frac{\partial U^{(1)}}{\partial y} + \xi \eta \frac{\partial W^{(1)}}{\partial y} + \xi \left(\frac{1}{2} \xi^2 - \frac{5}{2} \right) \frac{\partial \Theta^{(1)}}{\partial y} \right\}. \quad (16) \end{aligned}$$

This result, together with the expression (8) for $F^{(1)}$, is closely related to the Chapman-Enskog approximation to the distribution function (Chapman & Cowling 1961). The Chapman-Enskog theory assumes $\delta \ll 1$ but places no restrictions on ϵ (δ is effectively the Knudsen number for the model gas). N , \mathbf{U} , and Θ are not formally expanded but P_v , P_t , Q and all time derivatives are expanded in a series in δ which requires use of the conservation laws to remove all time derivatives. The first two terms in the expansion applied to the Krook equation are

$$\begin{aligned} F = \Psi - \frac{\delta}{N\Omega} \Psi \left\{ [(\xi - U)^2 - \frac{1}{3}(\xi - \mathbf{U})^2] \frac{1}{\Theta} \frac{\partial U}{\partial y} \right. \\ \left. + \frac{(\xi - U)(\eta - W)}{\Theta} \frac{\partial W}{\partial y} + \frac{\xi - U}{\Theta} \left[\frac{1}{2} \frac{(\xi - \mathbf{U})^2}{\Theta} - \frac{5}{2} \right] \frac{\partial \Theta}{\partial y} \right\} + O(\delta^2). \quad (17) \end{aligned}$$

If (17) is now expanded in a series in ϵ with terms of $O(\epsilon^2)$ retained; then the result obtained differs from

$$(2\pi)^{-\frac{1}{2}} \exp \left(-\frac{1}{2} \xi^2 \right) + \epsilon F^{(1)} + \epsilon^2 F^{(2)}$$

in two respects. First, there is no description of the relaxation of the initial conditions to near local thermodynamic equilibrium. In addition, the second-order contribution of Ψ is $\Psi^{(2)}$ and not $\Psi_M^{(2)}$. These are both consequences of the fact that the Chapman-Enskog procedure cannot describe the evolution of the motion from an arbitrary initial state. Thus, that part of $U^{(2)}$, $N^{(2)}$ and $\Theta^{(2)}$ that does not evolve solely on macroscopic time scales is not accounted for.

The result (16) may now be used to evaluate the ϵ^2 contributions to pressure tensor and heat flux vector. They are

$$\left. \begin{aligned} P_\nu^{(2)} &= e^{-s} \int \xi^2 \mathcal{F}^{(2)} d^3\xi + N^{(2)} + \Theta_M^{(2)} + N^{(1)}\Theta^{(1)} - \frac{4}{3\mathcal{R}} \frac{\partial U^{(1)}}{\partial y}, \\ P_t^{(2)} &= e^{-s} \int \mathcal{F}^{(2)} \xi \eta d^3\xi - \frac{1}{\mathcal{R}} \frac{\partial W^{(1)}}{\partial y}, \\ Q^{(2)} &= e^{-s} \int \left(\frac{1}{2} \xi^2 \xi \mathcal{F}^{(2)} - \mathbf{U}^{(1)} \cdot \xi \xi \mathcal{F}^{(1)} \right) d^3\xi - \frac{5}{2\mathcal{R}} \frac{\partial \Theta^{(1)}}{\partial y}. \end{aligned} \right\} \quad (18)$$

At this point the role of the parameter \mathcal{R} as the Reynolds number is clear. Consider the part of $P_t^{(2)}$ that survives after a few collision times

$$P_t^{(2)} = -\frac{\tau^*}{n_0 k T_0 \epsilon^2} = -\frac{1}{\mathcal{R}} \frac{\partial W^{(1)}}{\partial y} = -\frac{1}{\mathcal{R}} \frac{1}{\epsilon} \left(\frac{m}{k T_0} \right)^{\frac{1}{2}} L \frac{\partial W}{\partial x}.$$

Equating the shearing stress τ^* to $\mu \partial W / \partial x$ gives

$$\mathcal{R} = \epsilon (k T_0 / m)^{\frac{1}{2}} m n_0 L / \mu.$$

But $\epsilon (k T_0 / m)^{\frac{1}{2}}$ is the magnitude of the macroscopic velocity, $m n_0$ is the ambient density, and $L = l \delta^{-1}$ the length scale of the initial disturbance. Hence \mathcal{R} is a Reynolds number. An expression for μ is obtained by noting that

$$\mathcal{R} \equiv (\epsilon / \delta) = \epsilon L [n_0 \kappa(T_0)] (m / k T_0)^{\frac{1}{2}}.$$

Equating the above expressions for \mathcal{R} gives

$$\mu(T_0) = k T_0 / \kappa(T_0).$$

This expression is often used to obtain $\kappa(T_0)$ in terms of a given $\mu(T_0)$. An examination of $P_\nu^{(2)}$ gives the same expression for μ (as it must) and no bulk viscosity, while equating the macroscopic part of the heat flux vector to $-\lambda(\partial T / \partial x)$ (where λ is the thermal conductivity) reveals

$$P_r \equiv [\mu(T_0) / \lambda(T_0)] \frac{5}{2} k / m = 1.$$

This is the well-known property of the Krook model gas; a Prandtl number one for any form of $\kappa(T_0)$.

5. The uniformly valid macroscopic behaviour

Now consider the ϵ^3 terms in the conservation equations:

$$\left. \begin{aligned} \frac{\partial N^{(3)}}{\partial s} + \frac{1}{\mathcal{R}} \left\{ \frac{\partial N^{(2)}}{\partial \phi} + \frac{\partial N^{(1)}}{\partial \tau} + \frac{\partial}{\partial y} (U^{(2)} + N^{(1)} U^{(1)}) \right\} &= 0, \\ \frac{\partial U^{(3)}}{\partial s} + \frac{1}{\mathcal{R}} \left\{ \frac{\partial U^{(2)}}{\partial \phi} + \frac{\partial P_\nu^{(2)}}{\partial y} + \frac{\partial U^{(1)}}{\partial \tau} + U^{(1)} \frac{\partial U^{(1)}}{\partial y} + N^{(1)} \left(\mathcal{R} \frac{\partial U^{(2)}}{\partial s} + \frac{\partial U^{(1)}}{\partial \phi} \right) \right\} &= 0, \\ \frac{\partial W^{(3)}}{\partial s} + \frac{1}{\mathcal{R}} \left\{ \frac{\partial W^{(2)}}{\partial \phi} + \frac{\partial P_t^{(2)}}{\partial y} + \frac{\partial W^{(1)}}{\partial \tau} + U^{(1)} \frac{\partial W^{(1)}}{\partial y} + \mathcal{R} N^{(1)} \frac{\partial W^{(2)}}{\partial s} \right\} &= 0, \\ \frac{3}{2} \left(\frac{\partial \Theta^{(3)}}{\partial s} + N^{(1)} \frac{\partial \Theta^{(2)}}{\partial s} \right) + \frac{1}{\mathcal{R}} \left\{ \frac{3}{2} \left[\frac{\partial \Theta^{(2)}}{\partial \phi} + N^{(1)} \frac{\partial \Theta^{(1)}}{\partial \phi} + \frac{\partial \Theta^{(1)}}{\partial \tau} + U^{(1)} \frac{\partial \Theta^{(1)}}{\partial y} \right] \right. \\ \left. + \frac{\partial Q^{(2)}}{\partial y} + \frac{\partial U^{(2)}}{\partial y} + P_\nu^{(1)} \frac{\partial U^{(1)}}{\partial y} + P_t^{(1)} \frac{\partial W^{(1)}}{\partial y} \right\} &= 0. \end{aligned} \right\} \quad (19)$$

Substituting (13) and (18) into (19), and suppressing secular terms in s yields the following system:

$$\left. \begin{aligned} \frac{\partial N^{(2)}}{\partial \phi} + \frac{\partial U_M^{(2)}}{\partial y} + \frac{\partial N^{(1)}}{\partial \tau} + \frac{\partial}{\partial y} (N^{(1)}U^{(1)}) &= 0, \\ \frac{\partial U_M^{(2)}}{\partial \phi} + \frac{\partial}{\partial y} (N^{(2)} + \Theta_M^{(2)}) + \frac{\partial U^{(1)}}{\partial \tau} + U^{(1)} \frac{\partial U^{(1)}}{\partial y} + N^{(1)} \frac{\partial U^{(1)}}{\partial \phi} \\ &+ \frac{\partial}{\partial y} (N^{(1)}\Theta^{(1)}) - \frac{4}{3\mathcal{R}} \frac{\partial^2 U^{(1)}}{\partial y^2} = 0, \\ \frac{\partial W_M^{(2)}}{\partial \phi} + U^{(1)} \frac{\partial W^{(1)}}{\partial y} + \frac{\partial W^{(1)}}{\partial \tau} - \frac{1}{\mathcal{R}} \frac{\partial^2 W^{(1)}}{\partial y^2} &= 0, \\ \frac{3}{2} \frac{\partial \Theta_M^{(2)}}{\partial \phi} + \frac{\partial U_M^{(2)}}{\partial y} + \frac{3}{2} \left[N^{(1)} \frac{\partial \Theta^{(1)}}{\partial \phi} + \frac{\partial \Theta^{(1)}}{\partial \tau} + U^{(1)} \frac{\partial \Theta^{(1)}}{\partial y} \right] \\ &+ (N^{(1)} + \Theta^{(1)}) \frac{\partial U^{(1)}}{\partial y} - \frac{5}{2\mathcal{R}} \frac{\partial^2 \Theta^{(1)}}{\partial y^2} = 0. \end{aligned} \right\} \quad (20)$$

Equations (20) are exactly what would be obtained from the Navier–Stokes equations if all dependence on the kinetic time scale variable s were omitted and $(U_M^{(2)}, \Theta_M^{(2)})$ were replaced by $(U^{(2)}, \Theta^{(2)})$. It is, of course, necessary to put the specific heat ratio γ equal to $\frac{5}{3}$, and P_r equal to unity.

Up to this point only the kinetic time scale has been studied when suppressing the secular terms in the solutions. The elimination of such terms from the solutions to (20) will now determine $N^{(1)}$, $U^{(1)}$, and $\Theta^{(1)}$ uniquely for all times. First consider the third of (20). The equation can formally be integrated as

$$W_M^{(2)} = \int_0^\phi U^{(1)} d\phi \frac{\partial W^{(1)}}{\partial y}(\tau, y) + \phi \left\{ \frac{\partial W^{(1)}}{\partial \tau} - \frac{1}{\mathcal{R}} \frac{\partial^2 W^{(1)}}{\partial y^2} \right\} + W_M^{(2)}(\tau, y). \quad (21)$$

If $U^{(1)}(0, 0, y)$ vanishes sufficiently rapidly $y \rightarrow \pm \infty$, then

$$\int_0^{\pm \infty} U^{(1)}(0, 0, y) dy$$

is finite. Equations (12) then ensure that the first term in (21) is finite as $\phi \rightarrow \infty$. Hence, for $W_M^{(2)}$ to be bounded for all ϕ ,

$$\frac{\partial W^{(1)}}{\partial \tau} - \frac{1}{\mathcal{R}} \frac{\partial^2 W^{(1)}}{\partial y^2} = 0. \quad (22)$$

The transverse or shear velocity simply diffuses on the viscous time scale τ/\mathcal{R} . The uniformly valid first approximation is then

$$W^{(1)} = \int_{-\infty}^{\infty} dz W^{(1)}(0, 0, z) \exp[-\mathcal{R}(z-y)^2/4\tau] (\mathcal{R}/2\tau\pi)^{\frac{1}{2}}. \quad (23)$$

We next eliminate $\partial U_M^{(2)}/\partial y$ between the first and last of (20),

$$\begin{aligned} \frac{3}{2} \frac{\partial M^{(2)}}{\partial \phi} - \frac{\partial N^{(2)}}{\partial \phi} - \left[\frac{\partial N^{(1)}}{\partial \tau} + \frac{\partial}{\partial y} (N^{(1)}U^{(1)}) \right] + \frac{3}{2} \left(N^{(1)} \frac{\partial \Theta^{(1)}}{\partial \phi} + \frac{\partial \Theta^{(1)}}{\partial \tau} + U^{(1)} \frac{\partial \Theta^{(1)}}{\partial y} \right) \\ + (N^{(1)} + \Theta^{(1)}) \frac{\partial U^{(1)}}{\partial y} - \frac{5}{2\mathcal{R}} \frac{\partial^2 \Theta^{(1)}}{\partial y^2} = 0. \end{aligned} \quad (24)$$

Both $N^{(1)}$ and $\Theta^{(1)}$ contain terms proportional to $H^{(1)}(\tau, y)$. $H^{(1)}$ plays the same role in (24) that $W^{(1)}$ did in (21). Applying essentially the same arguments yields

$$\frac{\partial H^{(1)}}{\partial \tau} - \frac{1}{\mathcal{R}} \frac{\partial^2 H^{(1)}}{\partial y^2} = 0. \quad (25)$$

The non-isentropic part of the longitudinal motion is purely diffusive. The diffusion coefficient is $(\mathcal{R}P_r)^{-1}$ for an analysis starting from the Navier–Stokes equations. The solution to (25) is

$$H^{(1)} = \int_{-\infty}^{\infty} \left[\frac{3}{2} \Theta^{(1)}(0, 0, z) - N^{(1)}(0, 0, z) \right] (\mathcal{R}/2\pi\tau)^{\frac{1}{2}} \exp[-\mathcal{R}(z-y)^2/4\tau]. \quad (26)$$

In order to determine $J^{(1)}(r, \tau)$ and $G^{(1)}(\sigma, \tau)$ it is convenient to have (20) in characteristic form. Multiplying the continuity equation in (20) by $\frac{3}{5}$, the energy equation in (20) by $\frac{2}{5}$ and adding, we have

$$\begin{aligned} \frac{3}{5} \frac{\partial}{\partial \phi} (N^{(2)} + \Theta_M^{(2)}) + \frac{\partial U_M^{(2)}}{\partial y} + \frac{3}{5} \left[\frac{\partial N^{(1)}}{\partial \tau} + \frac{\partial}{\partial y} (N^{(1)}U^{(1)}) \right] \\ + \frac{3}{5} \left[N^{(1)} \frac{\partial \Theta^{(1)}}{\partial \phi} + \frac{\partial \Theta^{(1)}}{\partial \tau} + U^{(1)} \frac{\partial \Theta^{(1)}}{\partial y} \right] + \frac{2}{5} (N^{(1)} + \Theta^{(1)}) \frac{\partial U^{(1)}}{\partial y} - \frac{1}{\mathcal{R}} \frac{\partial^2 \Theta^{(1)}}{\partial y^2} = 0. \quad (27) \end{aligned}$$

Next multiplying the longitudinal momentum equation in (20) by $(\frac{3}{5})^{\frac{1}{2}}$ and adding this to (27), we get

$$\begin{aligned} \left[\left(\frac{3}{5} \right)^{\frac{1}{2}} \frac{\partial}{\partial \phi} + \frac{\partial}{\partial y} \right] \left[U_M^{(2)} + \left(\frac{3}{5} \right)^{\frac{1}{2}} (N^{(2)} + \Theta_M^{(2)}) \right] + \left(\frac{3}{5} \right)^{\frac{1}{2}} \left[\frac{\partial U^{(1)}}{\partial \tau} + U^{(1)} \frac{\partial U^{(1)}}{\partial y} + N^{(1)} \frac{\partial U^{(1)}}{\partial \phi} \right. \\ \left. + \frac{\partial}{\partial y} (N^{(1)}\Theta^{(1)}) - \frac{4}{3\mathcal{R}} \frac{\partial^2 U^{(1)}}{\partial y^2} \right] + \frac{3}{5} \left[\frac{\partial N^{(1)}}{\partial \tau} + N^{(1)} \frac{\partial \Theta^{(1)}}{\partial \phi} + \frac{\partial \Theta^{(1)}}{\partial \tau} + U^{(1)} \frac{\partial \Theta^{(1)}}{\partial y} + \frac{\partial}{\partial y} (N^{(1)}U^{(1)}) \right] \\ + \frac{2}{5} (N^{(1)} + \Theta^{(1)}) \frac{\partial U^{(1)}}{\partial y} - \frac{1}{\mathcal{R}} \frac{\partial^2 \Theta^{(1)}}{\partial y^2} = 0. \quad (28) \end{aligned}$$

We now consider a transformation of variables, replacing (ϕ, y) by (r, σ) ,

$$y = \frac{1}{2}(r + \sigma); \quad \phi = \frac{1}{2} \left(\frac{3}{5} \right)^{\frac{1}{2}} (\sigma - r).$$

When (12) is substituted into (28) it takes the form

$$2 \frac{\partial}{\partial \sigma} \left[U_M^{(2)} + \left(\frac{3}{5} \right)^{\frac{1}{2}} (N^{(2)} + \Theta_M^{(2)}) \right] - \left(\frac{5}{3} \right)^{\frac{1}{2}} \left\{ \frac{3}{5} \frac{\partial J^{(1)}}{\partial \tau} + \frac{2}{5} J^{(1)} \frac{\partial J^{(1)}}{\partial r} - \frac{3}{5\mathcal{R}} \frac{\partial^2 J^{(1)}}{\partial r^2} \right\} + B(\sigma, r, \tau) = 0.$$

Provided that $N^{(1)}$ and $\Theta^{(1)}$ are bounded initially in the same sense as $U^{(1)}$, then

$$\int_0^\sigma B(\sigma, r, \tau) d\sigma$$

is finite for all σ . The quantities $U_M^{(2)}$, $\Theta_M^{(2)}$ and $N^{(2)}$ can then be bounded for all σ only if

$$\frac{\partial J^{(1)}}{\partial \tau} + \frac{2}{5} J^{(1)} \frac{\partial J^{(1)}}{\partial r} - \frac{1}{\mathcal{R}} \frac{\partial^2 J^{(1)}}{\partial r^2} = 0. \quad (29)$$

This is, of course, Burgers' equation, specialized to the case $\gamma = \frac{5}{3}$, $P_r = 1$, and no bulk viscosity. The same procedure may be used to show that

$$\frac{\partial G^{(1)}}{\partial \tau} - \frac{2}{3} G^{(1)} \frac{\partial G^{(1)}}{\partial \sigma} - \frac{1}{\mathcal{R}} \frac{\partial^2 G^{(1)}}{\partial \sigma^2} = 0. \quad (30)$$

The solution and application of Burgers' equation to problems in which either $J^{(1)}$ or $G^{(1)}$ was present separately has been discussed by Lighthill (1956), particularly when \mathcal{R} is large. (Note that (30) becomes identical with (29) when σ is replaced by $-\sigma$.) The solutions to the initial-value problem are given by

$$\begin{aligned} J^{(1)}(r, \tau) &= (3/\mathcal{R}) \partial (\log R(r, \tau)) / \partial r, \\ R(r, \tau) &= \int_{-\infty}^{\infty} R(z, 0) (\mathcal{R}/2\pi\tau)^{\frac{1}{2}} \exp[-\mathcal{R}(z-y)^2/4\tau] dz, \\ R(r, 0) &= \exp \left\{ \frac{\mathcal{R}}{3} \int_0^r dz [U^{(1)}(0, 0, z) + (\frac{2}{3})^{\frac{1}{2}} (N^{(1)}(0, 0, z) + \Theta^{(1)}(0, 0, z))] \right\}, \\ G^{(1)}(\sigma, \tau) &= -(3/\mathcal{R}) \partial \log L(\sigma, \tau) / \partial \sigma, \\ L(\sigma, \tau) &= \int_{-\infty}^{\infty} L(z, 0) (\mathcal{R}/2\pi\tau)^{\frac{1}{2}} \exp[-\mathcal{R}(z-y)^2/4\tau] dz, \\ L(\sigma, 0) &= \exp \left\{ -\frac{\mathcal{R}}{3} \int_0^{\sigma} dz [(\frac{5}{3})^{\frac{1}{2}} (N^{(1)}(0, 0, z) + \Theta^{(1)}(0, 0, z)) - U^{(1)}(0, 0, z)] \right\}. \end{aligned}$$

The uniformly valid solution may thus be briefly described as follows. The distribution function relaxes to near local thermodynamic equilibrium after a few collision times; the departure from local equilibrium then being described correctly by the Chapman-Enskog procedure. The first approximation to the longitudinal motion obeys the acoustic equations initially. However, the motion eventually breaks into two propagating non-linear waves which ultimately decay through the action of viscosity and thermal conductivity, and an 'entropy' disturbance which diffuses very slowly due to the conductivity of the gas. The transverse motion diffuses over the same time scale, this time owing to viscosity.

The analysis presented here can be applied directly to the Navier-Stokes equations; it can be applied to flows in more than one dimension, and to a flow with boundaries. These topics will be considered in subsequent papers.

Finally, it should be pointed out (if it is not already obvious) that no attempt has been made to obtain a result that is uniformly valid for large values of ξ . There are two reasons for this. First, it is clear, from the form of the expressions obtained for F , that the macroscopic consequences of any such result would imply corrections of order $e^{-1/\epsilon}$. Second, the Krook model was set up specifically to focus on the macroscopic quantities N , U , Θ and makes no pretence of giving an accurate representation of F in velocity space. Hence it was assumed throughout that interest is confined to finite values of ξ .

This work was supported in part by the National Science Foundation under Grant NSF-GK-1827, and by the Division of Engineering and Applied Physics, Harvard University.

Appendix

The final determination of $F^{(1)}$ comes from the suppression of secular terms in the equation for $F^{(3)}$

$$\frac{\partial F^{(3)}}{\partial s} + \frac{1}{\mathcal{R}} \left(\frac{\partial F^{(2)}}{\partial \phi} + \xi \frac{\partial F^{(2)}}{\partial y} \right) + \frac{1}{\mathcal{R}} \frac{\partial F^{(1)}}{\partial \tau} = \Psi^{(3)} - F^{(3)} + \Omega^{(1)}(\Psi^{(2)} - F^{(2)}) + \Omega^{(2)}(\Psi^{(1)} - F^{(1)}). \tag{A 1}$$

The manner in which each quantity depends upon s may be shown explicitly in a notation as follows:

$$\begin{aligned} F^{(1)} &= e^{-s} \mathcal{F}^{(1)} + \Psi^{(1)}, & F^{(2)} &= e^{-s} \mathcal{F}^{(2)} + F_M^{(2)}, \\ \Omega^{(2)} &= \omega_2 e^{-s} + \Omega_M^{(2)}, & \Psi^{(3)} &= e^{-s} \psi_3 + \Psi_M^{(3)}, \\ \Psi^{(2)} &= e^{-s} \psi_2 + \Psi_M^{(2)}. \end{aligned}$$

Substituting this into (A 1) and suppressing secular terms in s leads to an inhomogeneous equation for $\mathcal{F}^{(2)}$

$$\frac{\partial \mathcal{F}^{(2)}}{\partial \phi} + \xi \frac{\partial \mathcal{F}^{(2)}}{\partial y} + \mathcal{R} \Omega^{(1)} \mathcal{F}^{(2)} = \mathcal{R} \psi_3 - \frac{\partial \mathcal{F}^{(1)}}{\partial \tau} - \mathcal{R} \Omega_M^{(2)} \mathcal{F}^{(1)}. \tag{A 2}$$

In the present notation, the equation for $\mathcal{F}^{(1)}$ (equation (15) above) is

$$\frac{\partial \mathcal{F}^{(1)}}{\partial \phi} + \xi \frac{\partial \mathcal{F}^{(1)}}{\partial y} + \mathcal{R} \Omega^{(1)} \mathcal{F}^{(1)} = \psi_2.$$

The formal solution to this is given by:

$$\begin{aligned} \mathcal{F}^{(1)} &= A^{(1)}(y_0, \xi, \tau) \exp \left\{ - \int_0^\phi \mathcal{R} \Omega^{(1)}(y_0 + \xi \phi', \phi' \tau) d\phi' \right\} \\ &+ \exp \left\{ - \int_0^\phi \mathcal{R} \Omega^{(1)}(y_0 + \xi \phi', \phi', \tau) d\phi' \right\} \int_0^\phi d\phi_1 \psi(\phi_1) \exp \left\{ \int_0^{\phi_1} \Omega^{(1)} d\phi' \right\}. \end{aligned} \tag{A 3}$$

Now, remembering that $\Omega^{(1)} = N^{(1)} + (1/\kappa_0) (d\kappa_0/dT_0) \Theta^{(1)}$, the assumed form of the initial conditions ensures that

$$\int_0^\infty \Omega^{(1)} d\phi < \infty.$$

As a result, all the inhomogeneous terms in (A 2) vanish as ϕ approaches infinity along the characteristics of the partial differential equation. $F^{(2)}$ is then finite for all ϕ on each characteristic, and there is no secular behaviour to suppress. Hence $A^{(1)}$ may be taken to be independent of τ .

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